for the conditioned quasiprobability distribution. The field ξ_2 in other words, has a Gaussian distribution about the mean value

$$<\xi_2> = \xi_1 \frac{G^{(1)}(x_1 x_2)}{G^{(1)}(x_1 x_1)} \qquad (14.69)$$

with a dispersion proportional to $G^{(2)}(x_2 x_2)\{1 - |g(x_1 x_2)|^2\}$, which vanishes for x_2 near x_1 and tends to approach $G^{(2)}(x_2 x_2)$ as x_2 recedes from x_1. We shall examine these expressions more closely once we have illustrated the evaluation of the correlation functions on which they depend.

Lecture XV   ELEMENTARY MODELS OF LIGHT BEAMS

Since our results to this point have all been stated in fairly general terms, it may be of help to discuss an illustrative example or two. Let us consider, as a particularly simple example, a stationary light beam which may be thought of as a plane wave progressing along the positive y-axis. We shall allow the beam to have an arbitrary frequency bandwidth, but shall take it to have a specific polarization θ. The first order correlation function for the beam may then be evaluated as a sum over plane wave mode functions by means of Eq. (14.13). The index which labels the mode functions in this case may be taken to be k_y, the y-component of the propagation vector. (The other components vanish.) Since the values of k_y are densely distributed, when the size L of the quantization volume is large, the sum over k_y is equivalent to a one-dimensional integration

$$\sum_{k_y} \cdot \frac{L}{2\pi} \int dk_y \ldots .$$

When the mode functions given by Eq. (R 2.9) are substituted in Eq. (14.13) and the sum is replaced by an integral, we find

$$G^{(1)}(y_1 t_1, y_2 t_2) = \frac{\hbar c}{4\pi} \int_0^\infty \frac{<n_k>}{L^3} k \exp\{-i[k_y(y_1 - y_2) - \omega_k(t_1 - t_2)]\} dk_y, \qquad (15.1)$$

where $G^{(1)}$ is understood to be a correlation function for the field components in the direction θ, as in Eq. (4.21). Since the beam contains no backward travelling waves, (which would be represented by negative values of k_y,) we may write the integral equally well as one over the frequency variable $\omega_k = ck_y$. Then if we introduce the parameter

$$s = t_1 - t_2 + \frac{1}{c}(y_1 - y_2) \qquad (15.2)$$

to express the space-time interval which occurs as an argument, we may write

$$G^{(1)}(y_1 t_1, y_2 t_2) = \frac{1}{4\pi c} \int_0^\infty \frac{<n_k> \hbar \omega_k}{L^2} e^{i\omega_k s} d\omega_k . \qquad (15.3)$$

The expression $<n_k> \hbar \omega_k$, which occurs in the integrand of Eq. (15.3), is the average energy of excitation of the k-th mode. Let us assume, as an example, that our beam has a spectral profile of the Lorentz form by writing

$$\frac{<n_k> \hbar \omega_k}{cL^2} = \frac{2\gamma}{(\omega - \omega_0)^2 + \gamma^2} U. \qquad (15.4)$$

Here ω_0 is the central frequency, γ is the half-width at half height, and the constant U is a measure of the intensity of the beam. Since the frequency ω_0 is typically much larger than γ, only a very small numerical error is made in the integration over the spectral porfile if the lower limit $\omega = 0$ in Eq. (15.3) is replaced by $\omega = -\infty$. By making this approximation and letting $\omega' = \omega - \omega_0$ we find

$$G^{(1)}(y_1 t_1, y_2 t_2) = \frac{\gamma}{2\pi} U e^{i\omega_0 s} \int_{-\infty}^{\infty} \frac{e^{i\omega' s}}{\omega'^2 + \gamma^2} d\omega'. \qquad (15.5)$$

The singularities of the function

$$\frac{1}{\omega'^2 + \gamma^2} = \frac{1}{2i\gamma}\left\{\frac{1}{\omega' - i\gamma} - \frac{1}{\omega' + i\gamma}\right\} \qquad (15.6)$$

are a pair of simple poles lying at $\pm i\gamma$ in the complex ω'-plane. The integral in Eq. (15.5) can be written as a contour integral around a closed path in the ω'-plane in either of two simple ways, depending on the sign of the variable s. For $s > 0$ the contour may be closed by means of an infinite semicircle in the upper half plane (Im $\omega' > 0$); for $s < 0$ it may be closed by a semicircle in the lower half plane. Since the integrals along both semicircles vanish, we find by applying the residue theorem

$$\int_{-\infty}^{\infty} \frac{1}{2i\gamma}\left\{\frac{1}{\omega - i\gamma} - \frac{1}{\omega + i\gamma}\right\} e^{i\omega s} d\omega = 2\pi i \begin{cases} \dfrac{1}{2i\gamma} e^{-\gamma s}, & s > 0 \\[2mm] \dfrac{1}{2i\gamma} e^{\gamma s}, & s < 0. \end{cases}$$

$$(15.7)$$

The first order correlation function, according to Eq. (15.5), is therefore given by

$$G^{(1)}(y_1 t_1, y_2 t_2) = \frac{1}{2} U e^{i\omega_0 s - \gamma|s|} . \qquad (15.8)$$

The intensity of the field is found by letting $y_1 = y_2$ and $t_1 = t_2$. For these values of the coordinates, which correspond to $s = 0$, we have

$$G^{(1)}(y_1 t_1, y_1 t_1) = \frac{1}{2} U. \qquad (15.9)$$

This is the average of the squared magnitude of the complex field $E^{(+)}$. It is easy to see, if we recall the formulae of elementary electrodynamics, that the parameter U is equal to the average total of the electric and magnetic energy densities for the field.

The correlation function given by Eq. (15.8) shows that our light beam exhibits approximate first order coherence when its frequency band width γ is sufficiently small. Thus, when we have

$$\frac{1}{\gamma} \gg |s| = |t_1 - t_2 - \frac{1}{c}(y_1 - y_2)|, \qquad (15.10)$$

the factor $e^{-\gamma|s|}$ in Eq. (15.8) may be approximated by unity, and the remainder of the expression for the correlation function may be written in the appropriate factorized form. As an alternative way of discussing first order coherence we note that the normalized form of the correlation function is

$$g^{(1)}(y_1 t_1, y_2 t_2) = \frac{G^{(1)}(y_1 t_1, y_2 t_2)}{\{G^{(1)}(y_1 t_1, y_1 t_1) G^{(1)}(y_2 t_2, y_2 t_2)\}^{\frac{1}{2}}}$$

$$= \exp[i\omega_0 s - \gamma|s|]. \qquad (15.11)$$

This function indeed has absolute magnitude close to unity as long as $\gamma|s|$ is sufficiently small.

A good deal of attention has been directed experimentally to the problem of developing light sources with narrow line width. In the best of these sources of the ordinary gas discharge or chaotic variety γ is of the order of 10^9 cycles per second. In ordinary laboratory sources it is often of order 10^{11} cycles per second or larger. The corresponding coherence ranges are 30 cm. and .3 cm. respectively.

Although we have been discussing the way in which monochromaticity may imply coherence, it may be worth recalling that it is not a necessary condition even for first order coherence. The coherence condition only becomes linked to a requirement of monochromaticity when we restrict our consideration to stationary fields, as we noted in connection with Eq. (7.24). For the case of stationary laser beams, the range of first order coherence is determined by the spectral bandwidth just as for ordinary sources. For the case of gas lasers it is possible to reduce the band width γ to values of the order of 10^3 cycles per second without too much difficulty, and it seems possible to achieve frequency stabilization to within about 10 cycles per second over brief intervals. The coherence ranges corresponding to these band widths are 300 km, and 30,000 km. respectively.

Before we can calculate the second and higher order correlation functions for our light beam, we must specify its statistical nature somewhat further. It is at this point that the descriptions of beams generated by natural sources and those generated by coherent sources become qualitatively different. Let us assume that our source is of the usual chaotic variety. Then the higher order correlation functions may all be expressed as sums of products of first order correlation functions, as we have seen in Eq. (14.36). The spectral density function of our plane wave beam, in other words, completely determines the statistical properties of the field. In particular the delayed coincidence rate for counting pairs of photons is given by

$$G^{(2)}(y_1 t_1, y_2 t_2, y_2 t_2, y_1 t_1) = G^{(1)}(y_1 t_1, y_1 t_1)\, G^{(1)}(y_2 t_2, y_2 t_2) + |G^{(1)}(y_1 t_1, y_2 t_2)|^2$$

$$= G^{(1)}(y_1 t_1, y_1 t_1)\, G^{(1)}(y_2 t_2, y_2 t_2)\{1 + |g^{(1)}(y_1 t_1, y_2 t_2)|^2\}$$

$$= (\tfrac{1}{2} U)^2 \{1 + e^{-2\gamma|s|}\} . \qquad (15.12)$$

The presence of the term $e^{-2\gamma|s|}$ in this expression shows that the beam can never possess second order coherence. Furthermore when we plot the coincidence rate against s as in Fig. 15 we see that that term constitutes the "bump" on the Hanbury Brown-Twiss correlation curve, i.e. the deviation of the curve from the accidental or background coincidence rate. The experimental curve shown earlier in Fig. 10 corresponds to a curve of the form shown here after the resolution properties of the counter system have been folded in.
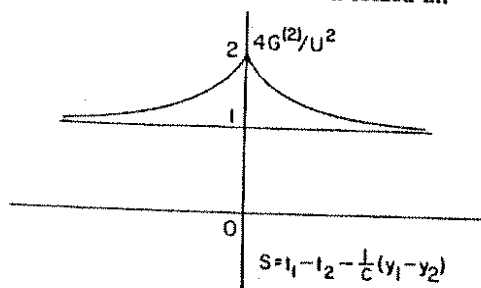

Figure 15

We have noted in the last lecture that the origin of the correlation effect lies in the random amplitude modulation of our light beam. Thus the factors of $n!$ by which the n-fold coincidence rate (at zero time delay) exceeds the random coincidence rate are easily explained in terms of the moments of the Gaussian amplitude distribution $W(\mathcal{E}, x)$ given by Eq. (14.48). To understand the behavior of the correlation effect for non-vanishing time delays, and to see, for example, why the effect disappears for $|s| \gg 1/2\gamma$, we may make use of the quasiprobability distributions defined for pairs of values of the field amplitude in the last lecture. When we substitute the values given by Eqs. (15.8) and (15.11) for the correlation functions into the expression (14.65) for the conditioned quasiprobability function $W(\mathcal{E}_1 x_1 | \mathcal{E}_2 x_2)$, we find

$$W(\mathcal{E}_1 y_1 t_1 | \mathcal{E}_2 y_2 t_2) = \frac{1}{\tfrac{1}{2}\pi U(1 - e^{-2\gamma|s|})} \exp\left\{-\frac{|\mathcal{E}_2 - \mathcal{E}_1 e^{i\omega_0 s - \gamma|s|}|^2}{\tfrac{1}{2}U\ (1 - e^{-2\gamma|s|})}\right\} . \qquad (15.13)$$

This function is to be interpreted as the distribution of values of the field amplitude $\mathcal{E}_2$ at $y_2 t_2$, when the amplitude is known to take on the value $\mathcal{E}_1$ at $y_1 t_1$. When the parameter s vanishes, the mean radius of the Gaussian peak of this expression vanishes and the distribution reduces to the delta function $\delta^{(2)}(\mathcal{E}_2 - \mathcal{E}_1)$. As $|s|$ increases from zero, the mean value of $\mathcal{E}_2$, which is given by $\mathcal{E}_1 e^{i\omega_0 s - \gamma|s|}$, describes an exponential spiral in the complex $\mathcal{E}_2$-plane while relaxing to the value zero. The spiral which corresponds to $s < 0$ is shown in exaggerated form in Fig. 16. At the same time the mean squared radius of the Gaussian peak of the
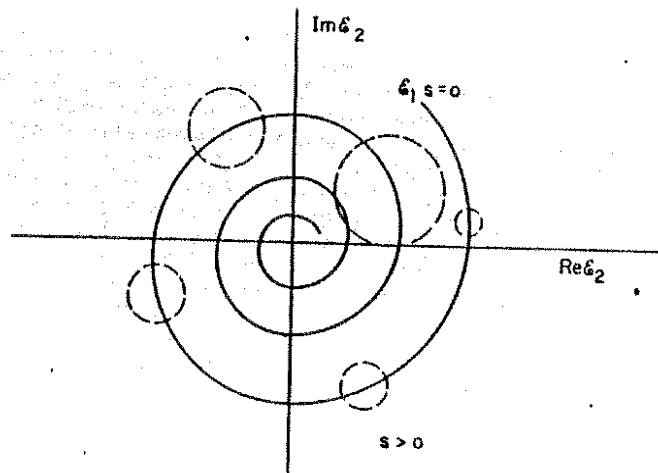

Figure 16

distribution increases to the asymptotic value $(1/2)U$. For values of $|s|$ much greater than $1/\gamma$ the conditioned distribution (15.13) relaxes to a form centered on the origin, which is simply the unconditioned distribution $W(\mathcal{E}_2, y_2 t_2)$ given by Eq. (14.48). The time $1/\gamma$ is a relaxation time for the field amplitude distributions. Our knowledge of $\mathcal{E}_1$ ceases to have much influence on the distribution of $\mathcal{E}_2$ for $|s| > 1/\gamma$. It is not surprising then that for intervals for which $|s| \gg 1/\gamma$ the two-photon coincidence rate, which is given by

$$G^{(2)}(y_1 t_1 y_2 t_2, y_2 t_2 y_1 t_1) = \int W(\mathcal{E}_1 y_1 t_1, \mathcal{E}_2 y_2 t_2) |\mathcal{E}_1|^2 |\mathcal{E}_2|^2 d^2\mathcal{E}_1 d^2\mathcal{E}_2$$

$$= \int W(\mathcal{E}_1 y_1 t_1) W(\mathcal{E}_1 y_1 t_1 | \mathcal{E}_2 y_2 t_2) |\mathcal{E}_1|^2 |\mathcal{E}_2|^2 d^2\mathcal{E}_1 d^2\mathcal{E}_2 ,$$

(15.14)

reduces to the factorized form

$$G^{(2)}(y_1 t_1 y_2 t_2, y_2 t_2 y_1 t_1) = G^{(1)}(y_1 t_1, y_1 t_1) G^{(1)}(y_2 t_2, y_2 t_2).$$

The tendency toward photon coincidences is wiped out, in other words, when the interval $s = t_1 - t_2 - c^{-1}(y_1 - y_2)$ becomes large because the field amplitudes $\mathcal{E}(y_1 t_1)$ and $\mathcal{E}(y_2 t_2)$ cease to be statistically correlated.

To see how the full time dependence of the coincidence rate emerges from the integral (15.14), we note that when the conditioned distribution function is given by Eq. (15.13), the average value of $|\mathcal{E}_2|^2$ when $\mathcal{E}_1$ is fixed is

$$\int W(\mathcal{E}_1 y_1 t_1 | \mathcal{E}_2 y_2 t_2) |\mathcal{E}_2|^2 d^2\mathcal{E}_2 = |\mathcal{E}_1|^2 e^{-2\gamma|s|} + \frac{1}{2} U(1 - e^{-2\gamma|s|}).$$

(15.16)

When this expression is multiplied by $|\mathcal{E}_1|^2$ and averaged, as in Eq. (15.14), over the Gaussian form for $W(\mathcal{E}_1 y_1 t_1)$, we find

$$G^{(2)}(y_1 t_1, y_2 t_2, y_2 t_2, y_1 t_1) = \left(\frac{1}{2} U\right)^2 \left\{ 2e^{-2\gamma|s|} + 1 - e^{-2\gamma|s|} \right\}$$

$$= \left(\frac{1}{2} U\right)^2 \left\{ 1 + e^{-2\gamma|s|} \right\},$$

(15.17)

which verifies the value of the coincidence rate found earlier in Eq. (15.12).

The values we have derived for the correlation functions have all been based on the assumption that the energy spectrum of our light beam has the Lorentz shape. The corresponding results are easily derived for other spectra for which the Fourier transform of the energy distribution is known. Other simple, smooth representations of the profile of a spectrum line, for example, lead to results which are qualitatively similar to those for the Lorentz line.

Since the photon correlation effect extends over delay times of the order of the inverse band width, γ, it might appear that this time can be stretched out by a factor of a million or more by using the extremely monochromatic light of the laser rather than light from natural sources. The error in such reasoning lies in the fact that the statistical properties of the laser beam are quite different from those of the chaotically generated beams we have been discussing. Lasers, when they are operating most monochromatically, generate beams with very little amplitude modulation, and for these, as we have seen in the last lecture, there would be virtually no photon correlation effect at all.

MODEL FOR IDEAL LASER FIELDS

For fields generated by chaotic sources, knowledge simply of the average occupation numbers $\langle n_k \rangle$ is sufficient to determine the density operator ρ, and from it all of the statistical properties of the field. However if our source is not chaotic in nature, we cannot expect that there will exist any self-evident way of finding the density operator for the field it generates without analyzing the mechanism by which it radiates in some detail. The only reliable method we have of constructing density operators, in general, is to devise theoretical models of the system under study and to integrate corresponding Schrödinger equation, or equivalently to solve the equation of motion for the density operator. These assignments are formidable ones for the case of the laser oscillator and have not been carried out to date in quantum mechanical terms. The greatest part of the difficulty lies

in the mathematical complications associated with the nonlinearity of the device. The nonlinearity plays an essential role in stabilizing the field generated by the laser. It seems unlikely, therefore, that we shall have a quantum mechanically consistent picture of the frequency bandwidth of the laser or of the fluctuations of its output until further progress is made with these problems.

If we are willing to overlook the noise and band width problems for the moment, and to confine our discussion to the case of an ideally monochromatic laser, then it is not difficult to find a representation for the density operator of the beam it generates. The radiation field is coupled within the laser to the electric dipole vectors of all of the atoms of the active medium. These atoms have a polarization which oscillates with the field and at the same time radiates energy into it. If we view the active medium as a whole, we see that it has an oscillating polarization density of macroscopic proportions, i.e., all neighboring atoms contribute similarly to the total polarization density. If we remember that the time derivative of a polarization density is, in effect, a current distribution, then we may think of the field as being radiated by the oscillating current distribution. When the laser is operating well above its threshold there is nothing weak about this current distribution; it is essentially of classical magnitude. Furthermore, if the laser has the ideal stability we have assumed, the current simply oscillates steadily in a perfectly predictable way. We may, in other words, to an excellent approximation, describe the bound current in the active medium as a c-number current density.

The general problem of finding the fields radiated by prescribed current distributions has been solved in Lecture XII. The most important property of the solution is that radiation by a known current distribution always brings the field to a coherent state (assuming that no other radiation was present initially). If the current oscillates with a single frequency, only the field modes with precisely that frequency will be excited. If we assume, for simplicity, that the geometry of our system favors the excitation of only one mode of the field, then the density operator for the field may be written in the form

$$\rho = |\alpha\rangle\langle\alpha|,$$

(15.18)

where $|\alpha\rangle$ is a coherent state for the excited mode, and the amplitude α is given by an integral of the form (12.20) taken over the bound current distribution.

Let us write the complex field eigenvalue which corresponds to the amplitude α as

$$\mathcal{E}(rt) = i\left(\frac{\hbar\omega}{2}\right)^{\frac{1}{2}} u(r) e^{-i\omega t} \alpha .$$

(15.19)

Then, since the density operator (15.18) corresponds to a pure coherent state, the correlation functions of all orders will factorize to the form of Eq. (8.5), i.e., the beam will possess full coherence. It follows then that the n-fold delayed coincidence rates will factorize to the form

$$G^{(2)}(x_1 \ldots x_n, x_n \ldots x_1) = \prod_{j=1}^{n} G^{(1)}(x_j, x_j),$$

(15.20)

and no photon coincidence correlations of any order will be detectable in the ideal laser beam.

The argument which led to the density operator (15.18) for the laser beam assumed that the oscillating current distribution is known precisely i.e., that we know its phase of oscillation as well as its amplitude. In practice our knowledge about quantities which oscillate at extremely high frequencies rarely includes any information about their absolute phase. (This is due more to the absence of a suitable clock to use as a reference standard than it is to any difficulty of principle in defining or measuring the phase of essentially classical quantities such as the

bound current in the laser.) When we lack any knowledge of the phase of oscillation of the current, the density operator should be written in an appropriately specialized form of Eq. (12.30). It is clear that this form is simply the expression (15.8) for the density operator averaged over the phase of the complex amplitude α, i.e.,

$$\rho = \int_0^{2\pi} |\,|\alpha|e^{i\theta}><|\alpha|e^{i\theta}|\,\frac{d\theta}{2\pi}$$

$$= \int \frac{1}{2\pi|\alpha|}\,\delta(|\beta| - |\alpha|)\,|\beta><\beta|\,d^2\beta . \tag{15.21}$$

These forms of the density operator depend on α only through its absolute value, and hence represent stationary fields. They represent mixed rather than pure states of the field, but as we have noted in the last lecture, mixtures corresponding to averaging an overall phase variable do not alter the coherence properties of the field. It is easy to verify that the correlation functions which are derived from the density operator (15.21) are identical to those which follow from (15.18).

The explicit construction of the density operator for an ideal laser beam shows that no photon correlations are to be detected in such a beam. The reason for the absence of such correlations is evident from the analysis of the last lecture. The quasiprobability function $W(\xi, x)$ which corresponds to the stationary density operator (15.21) is immediately seen from Eqs. (14.44) and (15.19) to be

$$W(\xi, x) = \frac{1}{2\pi\left(\frac{\hbar\omega}{2}\right)^{\frac{1}{2}}|u(r)\alpha|}\,\delta\left(|\xi| - \left(\frac{\hbar\omega}{2}\right)^{\frac{1}{2}}|u(r)\alpha|\right). \tag{15.22}$$

This function vanishes everywhere in the complex ξ-plane except on a circle where the delta function is singular. It describes a field which undergoes no amplitude modulation at all, and that is the basic reason for the absence of photon correlations in an ideal laser beam.

It is also possible, by making use of the correspondence principle, to see the origin of this property of coherently radiated beams more directly. We shall simplify our picture of the laser by regarding it simply as an oscillating charge distribution which radiates much as an antenna does. The charge, we assume, has only a single mode of vibration whose amplitude is, in effect, that of a harmonic oscillator. Since the electric polarization of this oscillator assumes macroscopic proportions we must regard the oscillator coordinate as an essentially classical quantity; i.e., the oscillator is typically in highly excited quantum states which have enormous quantum numbers.

When the oscillator is decoupled from whatever mechanism has excited it and allowed to radiate spontaneously, its amplitude of vibration will decrease quite slowly in relation to the oscillation period. Since the behavior of the oscillator is essentially classical, the current due to its moving charge distribution is quite predictable. As we have noted earlier, the radiation by such a current brings the field to a coherent state. If, on the other hand, we look at the oscillator from a quantum mechanical standpoint, we may think of it as making transitions downward in energy, step by step, passing through states with quantum numbers n, n − 1, n − 2 ··· where n ≫ 1. The length of time the oscillator spends in each of these states is distributed exponentially and, since n is so large, the average lifetimes of the states do not vary significantly from one state to the next. Each transition is accompanied by the emission of a photon. We are therefore not surprised to find that when the photons are detected by a counter, the intervals between their successive arrival times are exponentially distributed. This exponential distribution of time intervals indicates the absence of any tendency toward pair or higher order correlations. It is the characteristic distribution for the intervals between totally

uncorrelated events which happen at a fixed average rate. It is clear that where two or more counters are used there will be no time-dependent correlations of their outputs.

MODEL OF A LASER FIELD WITH FINITE BANDWIDTH

An actual laser beam, in contrast to the ideal variety we have just discussed, will never be precisely monochromatic. Its frequency is bound to vary more or less randomly over a narrow range due to disturbances which have their origin both inside and outside the laser itself. We shall construct a simple model of a laser field with finite frequency bandwidth by assuming that the mechanicsm which disturbs the laser is essentially stochastic in nature.

Let us assume, for simplicity, that the laser excites only a single mode of the electromagnetic field which has frequency ω_0. Then the field Hamiltonian for that mode is

$$H_0 = \hbar\omega_0 a^\dagger a$$

and, in the absence of any perturbing influences, the time-dependent operators $a(t)$ and $a^\dagger(t)$ are given in terms of the time-independent ones, a and $a^\dagger$, by

$$a(t) = a\,e^{-i\omega_0 t}$$

$$a^\dagger(t) = a^\dagger e^{i\omega_0 t} \tag{15.23}$$

The completely harmonic behavior of the oscillating field will be perturbed by various interactions of the field with other systems. We shall assume that the effect of these interactions can be represented by the addition of a term to the field Hamiltonian which depends on one or more random functions of time, $f(t)$. If we write this stochastic addition to the Hamiltonian as $H_f(t)$, the total field Hamiltonian becomes

$$H = H_0 + H_f(t). \tag{15.24}$$

To see the influence of the stochastic term most clearly we shall solve the Schrödinger equation in the interaction representation. The interaction Hamiltonian is then

$$H_f{}'(t) = e^{\frac{i}{\hbar}H_0 t}\,H_f(t)\,e^{-\frac{i}{\hbar}H_0 t} . \tag{15.25}$$

We define the unitary operator $U_f(t, t')$ as the solution of the Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}U_f(t, t') = H_f{}'(t)\,U_f(t, t') \tag{15.26}$$

which obeys the initial condition

$$U_f(t', t') = 1. \tag{15.27}$$

Then, if we write the state vector of the field at time t as $|t>$, we see that it evolves according to the transformation

$$|t> = U_f(t, t')\,|t'>.$$

The equation of motion for the density operator in the interaction representation, which we shall write as $\rho_i(t)$, is

$$i\hbar\frac{\partial}{\partial t}\rho_i(t) = [H_f{}'(t), \rho_i(t)]. \tag{15.28}$$

The solution for the time development of the density operator may be written in terms of the unitary operator U_t as

$$\rho_i(t) = U_i(t, t')\, \rho_i(t')\, U_i^{-1}(t, t'). \qquad (15.29)$$

The expressions for the field correlation functions which we have discussed earlier in these lectures have all been constructed according to Heisenberg picture of quantum mechanics in which the state vectors and the density operator are independent of time. When these vary with time, as in the interaction representation, the expectation values we require must be constructed somewhat differently. The required expressions can be found by starting with the form the expectation values take in the Heisenberg representation and carrying out the unitary transformation to the interaction representation.

Let us consider two arbitrary operators which take the time-dependent forms $L(t)$ and $M(t)$ in the Heisenberg representation. An example of the kind of statistical average which is used in the construction of the correlation functions is the averaged product which may be written as $< L(t) M(t') >_t$. The subscript on the average means that it is computed for a particular behavior of the random function $f(t)$ on which the stochastic Hamiltonian dependens. The average, when evaluated in the Heisenberg representation, is clearly

$$< L(t) M(t') >_t \quad = \mathrm{Tr}\{L(t) M(t')\rho\} \qquad (15.30)$$

where ρ is the time-independent Heisenberg density operator.

One of the ways of defining the Heisenberg representation (which is unitarily equivalent to all other ways) is to let the fixed Heisenberg state vector for the system be identical to the state vector in the interaction representation at a particular time t_0. Then the relation

$$|t> = U_i(t, t_0)\ |t_0> \qquad (15.31)$$

expresses the unitary transformation from Heisenberg states $|t_0>$ to states $|t>$ in the interaction representation. The corresponding transformations of the operators L, M and ρ are

$$L_i(t) \quad = U_i(t, t_0)\ L(t)\, U_i^{-1}(t, t_0)$$

$$M_i(t) = U_i(t, t_0)\ M(t)\ U_i^{-1}(t, t_0) \qquad (15.32)$$

$$\rho_i(t) = U_i(t, t_0)\, \rho\, U_i^{-1}(t, t_0),$$

where the subscripts i denote the forms of the operators in the interaction representation. When the inverted forms of these relations are used to express the operators in Eq. (15.30) we find

$$< L(t) M(t') >_t = \mathrm{Tr}\{U_i^{-1}(t, t_0)\, L_i(t)\, U_i(t, t_0)\, U_i^{-1}(t', t_0)\, M_i(t') \times$$

$$\rho_i(t')\, U_i(t', t_0)\} . \qquad (15.33)$$

Since the time displacement operator U_t obeys the multiplication law

$$U_i(t, t')\, U_i(t', t_0) = U_i(t, t_0), \qquad (15.34)$$

the expression for the average may be reduced to the form

$$< L(t) M(t') >_t = \mathrm{Tr}\{L_i(t)\, U_i(t, t')\, M_i(t')\rho_i(t')\, U_i^{-1}(t, t')\} . \qquad (15.35)$$

The occurrences of the operator U_t in this expression evidently take into account the effect of the disturbance of the field during the interval from t' to t. The disturbance, we are assuming, is a random one and the average (15.35) has been evaluated for some particular way in which it may behave, i. e., it is evaluated for a particular random function $f(t)$. Before the average can be compared with experiments it must again be averaged over a suitable ensemble of random functions $f(t)$. The latter averaging process is simplified by our use of the interaction representation.

Since the products LM which interest us are in normally ordered form it will be extremely convenient to make use of the P-representation for the density operator. We shall therefore only consider the class of stochastic Hamiltonians which preserve the possibility of expressing the density operator by means of the P-representation. We assume, in other words, that $\rho_i(t)$ may be written in the form

$$\rho_i(t) = \int P(\alpha, t)|\alpha><\alpha|\, d^2\alpha \qquad (15.36)$$

at all times t.

If the density operator at time t' corresponds to the pure coherent state $|\alpha>$, i. e.

$$\rho_i(t') = |\alpha><\alpha|, \qquad (15.37)$$

then, according to Eq. (15.29), at time t it will be

$$\rho_i(t) = U_i(t, t')\, \rho_i(t')\, U_i^{-1}(t, t')$$

$$= U_i(t, t')\, |\alpha><\alpha|U_i^{-1}(t, t'). \qquad (15.38)$$

Now, according to Eq. (15.36), this operator too will have a P-representation for which we may introduce the special notation

$$\rho_i(t) = \int P(\alpha t'|\beta t)\ |\beta><\beta|\, d^2\beta . \qquad (15.39)$$

The function $P(\alpha t'|\beta t)$ is evidently a conditioned quasiprobability function. It corresponds in the classical limit to a probability distribution for the complex amplitude β at time t, when we are given the knowledge that it had (or will have) the value α at time t'.

To illustrate the use of these relations in evaluating statistical averages, let us consider the average of the product $a^\dagger(t)\, a(t')$ which occurs in the first order correlation function. If we substitute $L(t) = a^\dagger(t)$ and $M(t) = a(t)$ into Eq. (15.35) we find, by using Eq. (15.23)

$$< a^\dagger(t) a(t') >_t = \mathrm{Tr}\{a^\dagger e^{i\omega_0 t}\, U_i(t, t')a\, e^{-i\omega_0 t'}\rho_i(t')U_i^{-1}(t, t')\} . \qquad (15.40)$$

Next we make use of Eq. (15.36) for the density operator, and the fact that $|\alpha>$ is an eigenstate of a to write

$$< a^\dagger(t) a(t') >_t = \mathrm{Tr}\{U_i(t, t')\int P(\alpha t')\alpha\, |\alpha><\alpha\,|d^2\alpha U_i^{-1}(t, t')a^\dagger\}e^{i\omega_0(t-t')} .$$

$$\qquad (15.41)$$

The unitary transformation inside the brackets may now be carried out by using Eq. (15.39) to represent the density operator indicated in Eq. (15.38). We then have

$$< a^\dagger(t) a(t') >_t = \mathrm{Tr}\{\int P(\alpha t')\alpha P(\alpha t'|\beta t)|\beta><\beta|\beta^* d^2\alpha\, d^2\beta\}e^{i\omega_0(t-t')}$$

$$= \int P(\alpha t')\, P(\alpha t'|\beta t)\, \alpha\beta^*\, d^2\alpha\, d^2\beta\, e^{i\omega_0(t-t')} . \qquad (15.42)$$

The latter expression for the average bears a close resemblance to forms which occur in the classical theory of continuous Markoff processes. We must now remember that the average we have constructed corresponds to some particular behavior of the random Hamiltonian. The quantity to be compared with experiment is not any one such value, but the average of all such values taken over a suitable ensemble of random functions $f(t)$. We may write this average as

$$< a^\dagger(t) a(t') > = \int < P(\alpha t') P(\alpha t \mid \beta t) >_{\substack{\text{av.} \\ \text{over } f}} \alpha\beta^* d^2\alpha d^2\beta e^{i\omega_0(t-t')} . \quad (15.43)$$

The foregoing equations furnish us with a fairly general framework for discussing the influence of random disturbances on the oscillations of the field. We shall now use this formalism in constructing a simple model of a laser beam of finite bandwidth.

Surely the simplest way to give the oscillating mode of the field a finite frequency bandwidth is to assume that its frequency is a random function of time. We may do this by writing the total field Hamiltonian of Eq. (15.24) as

$$H = \hbar[\omega_0 + f(t)] a^\dagger a , \quad (15.44)$$

where $f(t)$ is a random function of some sort whose ensemble average, $<f(t)>$, vanishes.

Since the random Hamiltonian is evidently

$$H_f(t) = \hbar f(t) a^\dagger a, \quad (15.45)$$

and it commutes with $H_0 = \hbar\omega a^\dagger a$, the interaction Hamiltonian according to Eq. (15.25) is simply H_f itself.

The Schrödinger equation (15.26) then takes the form

$$i\frac{\partial}{\partial t} U_f(t, t') = f(t) a^\dagger a U_f(t, t'). \quad (15.46)$$

Its solution is simply an exponential function which may be written in the form

$$U_f(t, t') = e^{-ia^\dagger a \phi(tt')} \quad (15.47)$$

where ϕ is defined by

$$\phi(tt') = \int_{t'}^{t} f(t'') dt''. \quad (15.48)$$

To see the effect of the transformation U_f on the states of the field, let us suppose that the field is in the coherent state $|\alpha>$ at time t'. Then at time t the state will be

$$|t> = U_f(t, t') |\alpha>$$
$$= e^{-ia^\dagger a \phi(tt')} |\alpha>$$
$$= e^{-\frac{1}{2}|\alpha|^2} e^{-ia^\dagger a \phi(tt')} \sum_{n=0}^{\infty} \{\alpha^n/(n!)^{\frac{1}{2}}\}|n>$$
$$= e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{(n!)^{\frac{1}{2}}} e^{-in\phi(tt')} |n> \quad (15.49)$$
$$= |\alpha e^{-i\phi(tt')} >.$$

The particular random Hamiltonian we have assumed just transforms one coherent state into another for which the amplitude parameter differs from the original one

by a phase factor. There is evidently no amplitude modulation in this model at all. When we use Eq. (15.46) to construct the density operator represented by Eqs. (15.38) and (15.39) we find

$$|\alpha e^{-i\phi(tt')}>< \alpha e^{-i\phi(tt')}| = \int P(\alpha t' \mid \beta t) |\beta> < \beta| d^2\beta, \quad (15.50)$$

from which we see that we may take the conditioned quasiprobability density to be simply the delta function

$$P(\alpha t' \mid \beta t) = \delta^{(2)}(\beta - \alpha e^{-i\phi(tt')}). \quad (15.51)$$

If we introduce the phases of the amplitudes α and β via the definitions

$$\alpha = |\alpha| e^{i\theta_0}$$
$$\beta = |\beta| e^{i\theta}, \quad (15.52)$$

then the two-dimensional delta function (15.51) can be written in terms of a product of two one-dimensional ones as

$$P(\alpha t' \mid \beta t) = \frac{1}{|\alpha|} \delta(|\beta| - |\alpha|) \delta(\theta - \theta_0 + \phi(tt')). \quad (15.53)$$

This function describes the evolution of the state of the field from the coherent state $|\alpha>$ at time t', when we are given any particular random function $f(t)$. To find the state at time t which is typical of the set of possible random functions, we must average Eq. (15.53) over the ensemble of functions $f(t)$. We may write this average as

$$P_{av} (\alpha t' \mid \beta t) = \frac{1}{|\alpha|} \delta(|\beta| - |\alpha|) <\delta(\theta - \theta_0 + \phi(tt'))>_{\text{av over } f}. \quad (15.54)$$

Now, if we recall that the function $\delta(\theta)$ has the Fourier series expansion

$$\delta(\theta) = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} e^{im\theta} , \quad (15.55)$$

we see that the averaged delta function in Eq. (15.54) may be written as

$$\delta(\theta) = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} e^{im(\theta-\theta_0)} <e^{im\int_{t'}^{t} f(t'')dt''} >_{\text{av over } f}. \quad (15.56)$$

We must clearly specify some of the properties of the random functions $f(t)$ before the exponential functions in Eq. (15.56) can be averaged over them.

The different physical processes which may perturb the frequency of our field oscillator require in general that we discuss various kinds of random functions $f(t)$. For the present, however, we shall only consider one of the simpler types of random functions. We shall assume that $f(t)$ is a stationary Gaussian stochastic process, i.e., that at any time t the ensemble of values of $f(t)$ has a fixed Gaussian distribution. Then it is not difficult to show that the averaged exponentials in Eq. (15.56) are given by

$$< \exp\{ im\int_{t'}^{t} f(t'')dt''\} >_{\substack{\text{av} \\ \text{over } f}} = \exp \{-\frac{1}{2} m^2 \int_{t'}^{t}\int_{t'}^{t} < f(t'')f(t''')> dt'' dt'''\}, \quad (15.57)$$

where the ensemble average $< f(t'')f(t''') >$ is simply the auto-correlation function of the random process $f(t)$.

Let us assume, simply as an illustration, that the function $f(t)$ fluctuates so

rapidly that its autocorrelation function can be taken to have the form

$$\langle f(l'') f(l''') \rangle = 2\zeta \, \delta(t'' - t''') \tag{15.58}$$

where ζ is a positive constant. Then the averaged exponential in Eq.(15.57) reduces to

$$\langle \exp\{im \int_{t'}^{t} f(t'') dt''\} \rangle_{aver \, f} = \exp\{-m^2 \zeta |t - t'|\}, \tag{15.59}$$

and the averaged delta function in Eq. (15.56) becomes

$$\langle \delta(\theta - \theta_0 + \phi(t')) \rangle_{aver \, f} = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} e^{im(\theta-\theta_0) - m^2 \zeta |t-t'|}. \tag{15.60}$$

It is interesting to note that this function is simply the Green's function of the partial differential equation for the diffusion of heat on a circular ring, i.e., it satisfies the equation

$$\left(\frac{\partial}{\partial t} - \zeta \frac{\partial^2}{\partial \theta^2}\right) \langle \delta(\theta - \theta_0 + \phi) \rangle_{av} = 0$$

for $t > t'$ and reduces to $\delta(\theta - \theta_0)$ for $t = t'$. It is clear then that the conditioned quasiprobability function (15.54), which we may write as

$$P_{av}(\alpha \, t' | \beta t) = \frac{1}{2\pi |\alpha|} \delta(|\beta| - |\alpha|) \sum_{m=-\infty}^{\infty} e^{im(\theta-\theta_0) - m^2 \zeta |t-t'|} \tag{15.61}$$

describes a kind of random phase modulation in which the phase variable $\theta = \arg \beta$ "diffuses" away from its initial value, θ_0.

The reciprocal of the diffusion constant ζ defines a relaxation time for the phase variable. For time intervals $t - t'$ which greatly exceed $1/\zeta$ the distribution (15.61) reduces to a constant, circularly symmetric form; the phase θ becomes completely random.

Let us now return to the question of evaluating the first order correlation function for the field. According to Eq. (15.43) we may construct the function as soon as we have evaluated the average

$$\langle P(\alpha, t') P(\alpha' t' | \beta t) \rangle_{aver \, f}. \tag{15.62}$$

We shall assume that we have no knowledge of the initial phase of oscillation of the field. Since the random perturbation of the field only shifts its phase, the phase remains uniformly distributed at all times; i.e., we never know more about the phase than we did initially. The density operator which represents the field is therefore stationary. The function $P(\alpha, t)$ in Eq. (15.36) depends on α only through its absolute value and is independent of t, and of the behavior of the function $f(t)$ as well. In this most frequently occurring case, the function $P(\alpha, t')$ may be written as $P(|\alpha|)$ and removed from the averaging brackets in the expression (15.62). That expression then reduces to the form

$$P(|\alpha|) \; P_{av}(\alpha t' | \beta t), \tag{15.63}$$

where the second factor is given by Eq. (15.61).

Now it is evident from Eq. (15.61) that

$$\int P_{av}(\alpha t' | \beta t) \, \beta^* d^2\beta = \frac{1}{2\pi |\alpha|} \int_0^\infty \delta(|\beta| - |\alpha|) |\beta|^2 d|\beta| \int_0^{2\pi} e^{-i\theta} \times$$

$$\sum_m e^{im(\theta-\theta_0) - m^2 \zeta |t-t'|} \, d\theta$$

$$= |\alpha| e^{-i\theta_0 - \zeta|t-t'|} = \alpha^* e^{-\zeta|t-t'|}. \tag{15.64}$$

On substituting the expression (15.63) into the correlation function (15.43) and making use of the integral just evaluated we find

$$\langle a^\dagger(t) a(t') \rangle = \int P(|\alpha|) |\alpha|^2 d^2\alpha \, e^{i\omega_0(t-t') - \zeta|t-t'|}$$

$$= \langle |\alpha|^2 \rangle \, e^{i\omega_0(t-t') - \zeta|t-t'|}, \tag{15.65}$$

where the symbol $\langle |\alpha|^2 \rangle$ has been used for the mean squared amplitude of excitation, or equivalently the average number of photons in the mode.

If we assume that the mode function $u(r)$ for the field does not change as a result of the perturbation, then the full space-time dependence of the first order correlation function may be found by multiplying the expression (15.65) by a product of the form $u^*(r) u(r')$. According to Eq. (R 10. 17), which is a quantum mechanical form of the Wiener-Khintchine theorem, the energy spectrum of the field will be porportional to the Fourier transform of the correlation function (15.65). When we calculate the transform we find

$$\int_{-\infty}^{\infty} \langle a^\dagger(0) a(t') \rangle e^{i\omega t'} dt' = \langle |\alpha|^2 \rangle \int_{-\infty}^{\infty} e^{i(\omega-\omega_0)t' - \zeta|t'|} dt'$$

$$= \langle |\alpha|^2 \rangle \frac{2\zeta}{(\omega - \omega_0)^2 + \zeta^2}. \tag{15.66}$$

Our phase diffusion model thus has an energy spectrum of Lorentzian shape, and the diffusion constant ζ is its half-width.

From a spectroscopic standpoint, the field we are describing could not be distinguished from the chaotically generated field of Lorentzian line shape which we discussed earlier, if we happened to have $\zeta = \gamma$. The fundamentally different nature of these two fields is best expressed by means of their higher order correlation functions. These functions may be evaluated for the phase diffusion model through simple extensions of the methods we have developed, but we shall not do so here. One fairly obvious result, however, is worth mentioning. Since the random phase modulation we have described carries no amplitude modulation with it, it will not introduce any photon coincidence correlations.

There are a number of ways in which the simple phase diffusion model which we have presented as an illustration can be generalized and made more realistic. We may easily remove, for example, the assumption that the stochastic process $f(t)$ has a vanishingly small relaxation time. Furthermore we may consider other types of stochastic processes than Gaussian ones. Finally, we may consider other forms of the random Hamiltonian than (15.45) and attempt in that way to account for some of the effects of random amplitude modulation as well as phase modulation.

Lecture XVI   INTERFERENCE
                        OF INDEPENDENT LIGHT BEAMS

One of the questions having to do with coherence which has given rise to much discussion and a certain amount of confusion recently is that of interference between independent light beams. That such interference phenomena can exist should come as no great surprise; they have been observed long ago with radio waves of fixed frequency. If we have had to wait until recently[1] to see such phenomena at optical frequencies, the delay has been wholly due to instrumental difficulties.

The problems which have arisen in the discussion of these interference phenomena concern the precise way in which they should be understood and described. It would be quite difficult to say how much of the misunderstanding we have mentioned is simply semantic in nature and how much is more deeply conceptual. There

is, for example, nothing intrinsically quantum mechanical about the interference of independent beams. Yet the fact that altogether different sets of quanta must somehow interfere with one another seems to have contributed greatly to the confusion. We shall not recount the history of this subject here but shall only discuss a few of the simplest possible examples of the interference phenomenon.

The simplest sort of experimental arrangement we can have is essentially that illustrated in Fig. 17. Two independent laser sources (or possibly other types of sources), L_1 and L_2 project their beams in directions which are nearly parallel, but slightly convergent. The beams fall upon overlapping areas of a screen Σ. If the light intensities are high enough, or we have sufficient time available to record over a long period, we may let our detector be a photographic film in the plane Σ. If the conditions do not favor photography, on the other hand, we might use a mosaic of photon counters in the plane Σ. In either case we will look for interference fringes in the area of overlap of the beams.
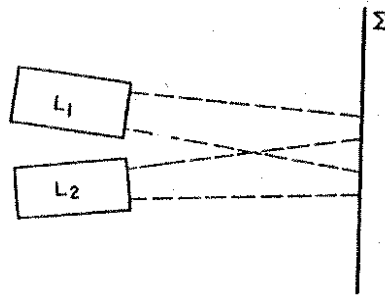

Figure 17

Let us assume that the way in which each light source excites the field can be described in the P-representation by means of functions $P_1(\{\alpha_{1k}\})$ and $P_2(\{\alpha_{2k}\})$. The single P-function which describes the superposed fields is then given, according to Eq. (R7.18) or (R9.15), by

$$P(\{\alpha_k\}) = \int P_1(\{\alpha_{1k}\}) P_2(\{\alpha_{2k}\}) \prod_k \delta^{(2)}(\alpha_k - \alpha_{1k} - \alpha_{2k}) \, d^2\alpha_{1k} \, d^2\alpha_{2k}.$$

(16.1)

The average intensity of the superposed fields at any space-time point x is given by the first order correlation function

$$G^{(1)}(x, x) = \int P(\{\alpha_k\}) |\mathscr{E}(x, \{\alpha_k\})|^2 \prod_k d^2\alpha_k$$

(16.2)

$$= \int P_1(\{\alpha_{1k}\}) P_2(\{\alpha_{2k}\}) |\mathscr{E}(x, \{\alpha_{1k} + \alpha_{2k}\})|^2 \prod_k d^2\alpha_{1k} d^2\alpha_{2k}.$$

In reaching the second of these expressions we have made use of Eq. (16.1) and have carried out the integrations over the variables $\{\alpha_k\}$. Now let us note that the eigenvalue field $\mathscr{E}(x, \{\alpha_k\})$ depends linearly upon the amplitudes α_k so that we have

$$\mathscr{E}(x, \{\alpha_{1k} + \alpha_{2k}\}) = \mathscr{E}(x, \{\alpha_{1k}\}) + \mathscr{E}(x, \{\alpha_{2k}\}),$$

(16.3)

a statement which corresponds to the classical superposition principle. If we substitute this relation in Eq. (16.2), and let the symbols $\{G^{(1)}(x, x)\}_j$ with $j = 1, 2$, be the intensities which would be produced by either source in the absence of the other, then we may write the total intensity as

$$G^{(1)}(x, x) = \{G^{(1)}(x, x)\}_1 + \{G^{(1)}(x, x)\}_2$$

$$+ 2\mathrm{Re}\left\{\int P_1(\{\alpha_{1k}\}) \mathscr{E}^*(x\{\alpha_{1k}\}) \prod_k d^2\alpha_{1k} \int P_2(\{\alpha_{2k}\}) \mathscr{E}(x\{\alpha_{2k}\}) \prod_k d^2\alpha_{2k}\right\}.$$

(16.4)

The third term of this sum is evidently an interference term. We must next ask when it contributes to the observed intensities and when it does not.

We have noted in Section VII of the reprinted paper that any light beam described in the P-representation can be regarded as the superposition of two fields, one of which corresponds to a pure coherent state and the other of which is of the unphased form, i.e., it has vanishing expectation value for the complex field strength. When each of the fields generated by the two sources is analyzed in this way, it becomes clear that the unphased components of the fields will not contribute to the interference term in Eq. (16.4). The interference term will, in fact, vanish completely unless the field generated by each of the two sources has a non-zero coherent component.

The most elementary kind of example in which the interference term is different from zero is one in which the two sources acting separately bring the field to coherent states represented by

$$P_1(\{\alpha_{1k}\}) = \prod_k \delta^{(2)}(\alpha_{1k} - \beta_{1k})$$

$$P_2(\{\alpha_{2k}\}) = \prod_k \delta^{(2)}(\alpha_{2k} - \beta_{2k}).$$

(16.5)

Then the interference term of Eq. (16.4) reduces to

$$2\mathrm{Re}\{\mathscr{E}^*(x, \{\beta_{1k}\}) \mathscr{E}(x, \{\beta_{2k}\})\}.$$

(16.6)

The analysis of this term may be simplified by assuming that the two sources are ideal lasers which are similar in construction and that each excites only a single plane wave mode. The two plane wave modes are then not identical since their propagation vectors are not quite parallel, but they have the same frequency. Under these conditions it is easy to see that the interference term (16.6) describes stationary intensity fringes which are seen on the screen in the area in which the two beams overlap. The fringes are perpendicular to the plane which contains the two propagation vectors and may be made narrow or broad by making the angle between the beams large or small.

Let us suppose that the single mode excited by source 1 has amplitude β_1 and that excited by source 2 has amplitude β_2. Then, since the plane wave mode functions are intrinsically complex, it is clear that the position of the fringe system on the screen Σ (i.e., its displacement in the direction perpendicular to the fringes) will depend on the phase difference of the complex amplitudes β_1 and β_2. If the geometry of the experiment is sufficiently well determined, then by observing the fringe system we may measure the phase difference.

No difficulty of principle stands in the way of our actually carrying out experiments of the type we have just described with two laser beams. But in practice we never have the complete knowledge of the excitation amplitudes which we assumed, for example, in constructing Eqs. (16.5) and (16.6). As we have remarked many times earlier, we are almost always lacking knowledge of overall phase parameters. As long as this is so we do not know the phases of oscillation of our lasers, and the only way we can honestly represent the density operators for the modes they excite is by means of the functions

$$P_j(\alpha_j) = \frac{1}{2\pi|\beta_j|} \delta(|\alpha_j| - |\beta_j|)$$

(16.7)

for $j = 1, 2$. These functions represent the stationary density operators which are

obtained, as in Eq. (15.21), by averaging the coherent states over phase. But the P-functions (16.7) are of the unphased variety; they correspond to vanishing averaged complex fields. When the descriptions of our two sources are stationary, in other words, the interference term in Eq. (16.4) vanishes identically.

If this result is taken to mean that there are no fringes to be seen on the screen, then our ignorance of the phase parameters has somehow wiped out a large scale physical phenomenon. To bring the paradox of such a conclusion into sharper focus it is possible to argue that each of our laser sources is essentially classical in nature and really has a well defined phase of oscillation. Consequently the fringes should be visible on the screen both to people who do and who don't know the phases alike.

To see that we have not really encountered any fundamental dilemma we must recall that density operators are constructed for the purpose of describing en-sembles of quantum mechanical experiments. The need to repeat experiments upon many similarly prepared systems arises for reasons which are quite basic to quantum mechanics. The quantities measured in general fluctuate unpredictably from one system to another, even when all the systems are prepared in precisely the same quantum state. When the quantum state itself is random there is still a further reason for carrying out experiments on a large number of systems and averaging their results.

The two P-functions given by Eq. (16.5) represent, for example, pure states of the field. In any single experiment carried out with two sources for which all the excitation amplitudes and phases are known, we would probably detect a more-or-less noisy form of the interference pattern we have been discussing. The inter-ference pattern would assume the smooth form given by Eq. (16.6) only after we had averaged over many experiments performed with identically prepared sources.

Now when we have no knowledge of the phases of oscillation of our two laser sources, our formalism describes an ensemble of experiments in which the phases are allowed to be completely random. It is true that the contribution of the inter-ference effect to the average intensity for this ensemble vanishes. But one can not conclude from the vanishing of the ensemble average that the fringes do not show up in the individual experiments. This experiment is one in which the mem-bers of the ensemble are individually quite unlike their ensemble average. Each of the experiments will exhibit a stationary fringe pattern on the screen, just as when the oscillation phases are known. But since the phases are random, the displace-ment of the pattern will vary randomly from one experiment to the next. It is the averaging over the random displacement which wipes away the fringes in the en-semble average.

A question we might now ask is how we can use the density operator formal-ism at all to make statistical statements about the fringe pattern. When the sources are stationary it has appeared to tell us nothing but that the ensemble average of the interference intensity vanishes at every point on the screen. Let us imagine that we are performing the experiment with a pair of lasers chosen from our random phase ensemble. To determine that there is indeed an interference pattern on the screen we must measure the intensity at a considerable number of points on the screen. We do not prepare the system anew for each of these mea-surements; they are carried out for a single preparation of the lasers. Now just the first of the intensity measurements at a known point on the screen goes a long way toward determining the phase difference of the two lasers. It determines a linear combination of the sine and cosine of the phase difference of the amplitudes β_1 and β_2 which restricts the phase difference to either of two discrete values. Measurment of the intensity at another point then determines the phase difference.

Once we have used intensity measurements at a couple of points to determine the phase difference we can predict the appearance of the rest of the interference pattern in an ensemble average sense. Of course the ensemble in this case is no longer the one we began with, though it still remains a stationary one. Our initial

intensity measurements furnish us with information which requires that we redu the size of our initial ensemble by retaining only those experiments in which the phase difference is found to be nearly the same. This reduced ensemble will be described by a stationary density operator since a phase factor common to the amplitudes β_1 and β_2 of a pair of degenerate modes remains completely random Let us suppose that we find the phase difference of the two beams to be

$$\arg \beta_1 - \arg \beta_2 = \theta.$$ (16.

Then the selection process by which we reduce the ensemble to one appropriate experiments for fixed θ can be represented by inserting a factor

$$\frac{1}{2\pi} \delta(\arg \alpha_1 - \arg \alpha_2 - \theta)$$ (16.

into the integrand of the P-function (16.1). Once we have located the fringe pat tern by experimentally determing its unpredictable position, we have no difficul in constructing a stationary density operator which predicts the average intensi in the pattern.

The idea of reducing the size of our ensemble to reflect the acquisition of knowledge about a system should not be too unfamiliar. In any multi-step gam chance, for example, the odds for winning, which one hopes are even initially, change as one completes each move. The initial odds are calculated by using th complete ensemble of possible games, but the odds calculated at the later states use only the reduced ensembles appropriate to the information which was reveal by the earlier moves.

Another sense, though a rather different one, in which the use of the station density operator furnishes information about the randomly placed interference pattern may be seen by discussing the second order correlation function. It is e to show that the two-fold coincidence counting rate

$$G^{(2)}(rt\ r't',\ r't'\ rt) = \int P(\{\alpha_k\})|\,\mathcal{E}(rt\{\alpha_k\})|^2|\,\mathcal{E}(r't\{\alpha_k\})|^2 \prod_k d^2\alpha_k$$

(16.

contains a term which oscillates as a function of the positions r and r' on the scr This type of interference term may be derived by means of essentially the same argument as we used in discussing the intensity interference experiments in Lecture II. The oscillation of the intensity correlation function must evidently r flect oscillation of the intensity itself. Furthermore since the unknown phase angles of β_1 and β_2 cancel out of the second order correlation function nothing need be known about them to calculate it.

However a simple measurement of the intensity of a random fringe pattern (e.g., by examining a photograph) is not the same as a measurement of $G^{(2)}$, a there is no simple way of concluding in general from a knowledge of $G^{(2)}$ what th intensity pattern of the random fringe system should be. Thus, while $G^{(2)}$ and tl other even order correlation functions are useful in their own right, they offer n alternative way of discussing the fringe intensities. If we want the intensities w must derive them from the density operators for appropriately reduced ensembl

We have assumed to this point that our light sources are ideal noise-free la We now ask what happens when the random modulation of the devices is taken int account. Since the most important of the parameters in determing the two-beam interference pattern is the phase of oscillation of the laser, we can secure a goo idea of what goes on by using the phase diffusion model to represent the laser beams. According to that model, the phase of a laser beam wanders appreciabl over time-intervals long compared to a relaxation time $1/\zeta$, and remains relati ly fixed over time intervals which are much shorter in length.

When the two laser beams are represented by such models, the light intensit

e record on the screen will depend on the length of time we require to make our measurements. If the intensities are sufficiently great that we can record them in time short compared to $1/\zeta$, then the two beams will retain nearly the initial values of their phases while the measurements are being made. A randomly situated fringe pattern of the sort we have already discussed should then show up. But a similar measurement made, say, half a relaxation time later would reveal a differently placed set of fringes, corresponding to the fluctuation that had taken place in the phase difference of the two beams.

If we could follow the fringe intensity as a function of time, we should see the parallel fringe system execute a sort of random wandering back and forth on the screen. If we were to try recording the intensities on the screen by integrating these over a period much longer than the relaxation time we would find that the fringe structure is washed out and only a uniform intensity remains.

Laser sources are convenient ones for such two-beam experiments, because they are intense, and monochromatic enough to have relatively long relaxation times. It is also quite possible, in principle, to carry out such experiments with beams from ordinary chaotic sources. The random amplitude modulation of these beams will mean that the fringes fluctuate greatly in contrast as well as in position. The relaxation time for these variations will be the inverse frequency bandwidth of the sources. If such fringes have not been photographed to date, it is because exposure times shorter than 10^{-10} sec. would be necessary.

REFERENCES

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Lecture XVII   PHOTON COUNTING EXPERIMENTS

The number of photons which a counter records in any interval of time fluctuates randomly. In a simple type of counting experiment we might imagine that the counter is exposed to the field for a fixed interval of time t. Then, by repeating the experiment many times, we should find a distribution function for the number of counts received in that interval. Although the average number of counts is frequently all that we require, the way in which the number fluctuates about its average value can be fully understood only when we know the distribution function and its moments. In this lecture we shall discuss ways of predicting the distribution function and the relation between the form of the distribution and the coherence of the field.

Let us first recall some of the results we established in Lecture V. We calculated there the probability that in an interval of time from t_0 to t all n atoms of a hypothetical n-atom photodetector undergo photoabsorption transitions which are registered as photon counts. When we eliminate the tensor indices by assuming the field to be fully polarized, this probability is given by Eq. (5.8), i.e., we have

$$p^{(n)}(t) = \int_0^t \cdots \int_0^t \prod_{j=1}^{n} S(t_j'' - t_j') \, G^{(n)}(r_1 t_1' \cdots r_n t_n', \, r_n t_n'' \cdots r_1 t_1'') \times$$

$$\prod_{j=1}^{n} dt_j' \, dt_j'', \qquad (17.1)$$

where the sensitivity function S is defined by Eqs. (4.12) and (4.10), and we have $t_0 = 0$. If our detector happens to be of the broadband variety, we may use Eq. (4.14) to reduce the number of time integrations in this integral from 2n to n, but

this reduction is not a necessary one for the arguments to follow.

We must now consider a more realistic model of a counter which contains an enormous number of atoms, say $N \sim 10^{20}$, which are capable of detecting photons by undergoing photoabsorption processes. Needless to say, it will virtually never happen that all N of these atoms do undergo absorption processes in any finite interval of time. The total number of photoabsorptions is much smaller as a rule, and we shall try to use Eq. (17.1) to find its distribution law.

The total number of photocounts recorded in any interval of time may be regarded as a sum of random variables, one for each atom of the detector. To do this, let us introduce the random variable z_j for the j-th atom, which takes on the values

$$z_j = \begin{cases} 0 \text{ if no photoabsorption process is recorded for the j-th atom} \\ 1 \text{ if a photoabsorption process is recorded for the j-th atom.} \end{cases} \qquad (17.2)$$

Then the random variable which represents the total number of counts will be

$$C = \sum_{j=1}^{n} z_j. \qquad (17.3)$$

Associated with each final state of the system i.e., any set of values $z_1 \cdots z_N$, there is a probability function $\mathcal{P}(z_1 \cdots z_N, t)$. The statistical average of any function of the z_j's is then found by averaging the function over the probability distribution. For example, the average number of counts is given by

$$\langle C \rangle = \sum_{\{z_j = 0\}} \sum_{j=1}^{N} z_j \, \mathcal{P}(z_1 \cdots z_N, t), \qquad (17.4)$$

where the final summation is over the values 0 and 1 for the entire set of variables z_j. We shall write such sums in the future as sums over $\{z_j\}$. We next introduce the reduced probability function for the j-th atom which we define as

$$p_j(z_j, t) = \sum_{\{z_k, k \neq j\}} \mathcal{P}(z_1 \cdots z_N, t). \qquad (17.5)$$

The average number of counts may be written in terms of the reduced probabilities p_j as

$$\langle C \rangle = \sum_{\{z_k\}} \sum_{j=1}^{N} z_j p_j(z_j, t)$$

$$= \sum_{j=1}^{N} p_j(1, t). \qquad (17.6)$$

The probability $p_j(1, t)$ which occurs in the latter expression is clearly equal to the one-atom transition probability $p^{(1)}(t)$ evaluated for the j-th atom. That probability is given by Eq. (17.1) for n = 1, with $r_1 = r_j$, and we shall write it as $p^{(1)}_j(t)$. The average number of counts is thus

$$\langle C \rangle = \sum_{j=1}^{N} p^{(1)}_j(t). \qquad (17.7)$$

We shall now introduce a generating function which will enable us to solve simultaneously for the unknown distribution of photocounts and for its moments. We could, of course, find the moments directly by generalizing the way in which $\langle C \rangle$ was obtained, but the present method has the advantage of enabling us to obtain all the quantities of interest from a single function. The generating function we choose is

$$Q(\lambda, t) = \langle (1 - \lambda)^C \rangle, \qquad (17.8)$$

ere C is the random integer given by Eq. (17.3), the brackets indicate an ensemble average, and the variable λ is intended simply to be a useful parameter.

If we write Q as a sum over the integer values which C may take on we have expansion of the form

$$Q(\lambda, t) = \sum_{m=0}^{N} (1 - \lambda)^m p(m, t), \qquad (17.9)$$

ere p(m, t) is the probability that the counter has recorded m photocounts at time t. It is clear that if $Q(\lambda, t)$ is known p(m, t) can be obtained by differ-iation,

$$p(m, t) = \frac{(-1)^m}{m!} \left[\frac{d^m}{d\lambda^m} Q(\lambda, t) \right]_{\lambda = 1}, \qquad (17.10)$$

ce Eq.(17.9) may be regarded as a Taylor expansion for Q about $\lambda = 1$.

If, on the other hand, we expand $Q(\lambda, t)$ in a power series about $\lambda = 0$ we have

$$Q(\lambda, t) = \sum_{n=0}^{N} \frac{\lambda^n}{n!} \left[\frac{d^n}{d\lambda^n} Q(\lambda, t) \right]_{\lambda = 0}. \qquad (17.11)$$

e derivatives which occur in this expansion are given by

$$(-1)^n \left[\frac{d^n}{d\lambda^n} Q(\lambda, t) \right]_{\lambda = 0} = \left\langle \frac{C!}{(C - n)!} \right\rangle$$
$$= < C(C - 1) \ldots (C - n + 1) >. \qquad (17.12)$$

e averages on the right of this equation are known as factorial moments. They e simple linear combinations of the ordinary moments $< C^n >$ of the distribution photocounts. It is clear from these relations that a knowledge of the generating iction enables us to find both the probability distribution and its moments. We ist next show how it is possible to evaluate the generating function in terms of photoabsorption probabilities $p^{(n)}(t)$.

First let us note that $Q(\lambda, t)$ can be written as

$$Q(\lambda, t) = \sum_{\{z_k\}} \mathcal{P}(z_1 \ldots z_N t) (1 - \lambda)^{\sum_{j=1}^{N} z_j}$$

$$= \sum_{\{z_k\}} \mathcal{P}(z_1 \cdots z_N) \prod_{j=1}^{N} (1 - \lambda)^{z_j}. \qquad (17.13)$$

e latter form, however, may be simplified by using the identity

$$(1 - \lambda)^{z_j} = 1 - z_j \lambda, \qquad (17.14)$$

lich holds because z_j takes on only the values zero and one. With this simpli-:ation, Eq. (17.13) becomes

$$Q(\lambda, t) = \sum_{\{z_j\}} \mathcal{P}(z_1 \ldots z_N, t) \prod_{j=1}^{N} (1 - \lambda z_j). \qquad (17.15)$$

hen the N-fold product in this expression is expanded in powers of λ, we have

$$Q(\lambda, t) = \sum_{n=0}^{N} (-\lambda)^n \sum_{\{z_j\}} \sum_{\substack{n\text{-fold} \\ \text{combinations}}} z_{j_1} \ldots z_{j_n} \mathcal{P}(z_1 \ldots z_N, t), \qquad (17.16)$$

where the first sum is taken over all the ways of choosing n atoms from the set of N.

If we now define the n-fold joint probability that atoms $j_1 \ldots j_n$ all undergo photoabsorption processes as

$$p^{(n)}_{j_1 \ldots j_n}(t) = \sum_{\{z_k\}} z_{j_1} \ldots z_{j_n} \mathcal{P}(z_1 \ldots z_N t), \qquad (17.17)$$

then we may write the generating function in the form

$$Q(\lambda, t) = \sum_{n=0}^{N} (-\lambda)^n \sum_{n\text{-fold combinations}} p^{(n)}_{j_1 \ldots j_n}(t). \qquad (17.18)$$

Now the number $p^{(n)}_{j_1 \ldots j_n}(t)$ has been defined as the probability that each of a particular set of n atoms absorbs a photon, regardless of what all the other atoms do. This probability is simply the expression $p^{(n)}(t)$ given by Eq. (17.1) and evaluated for the particular atoms $j_1 \ldots j_n$. Hence we know all the terms of Eq. (17.18) and the problem is simply to sum them. What we shall do, in fact, is to turn the sums over atoms into volume integrations.

Since the probabilities $p^{(n)}(t)$ are only large for values of n which are extremely small in comparison with N, we may approximate the sums over n-fold combinations by writing

$$\sum_{n\text{-fold combinations}} \approx \frac{1}{n!} \sum_{j_1=1}^{N} \sum_{j_2=1}^{N} \ldots \sum_{j_n=1}^{N}. \qquad (17.19)$$

Then the sums over the individual atoms may be carried out as spatial integrations by letting the number of atoms per unit volume be $\sigma(r)$ and writing

$$\sum_{j_1=1}^{N} \cdots = \int dr_1 \, \sigma(r_1) \ldots \qquad (17.20)$$

We are, in effect, dealing with the limit $N \to \infty$. When the probabilities given by Eq. (17.1) are substituted in the expression (17.18) for the generating function and the sum over combinations of atoms is transformed as we have indicated, we find

$$Q(\lambda, t) = \sum_{n=1}^{\infty} \frac{(-\lambda)^n}{n!} \int_{t_0}^{t} \ldots \int_{t_0}^{t} \int_{\text{Vol. of Detector}} \cdots \int_{\text{Vol. of Detector}}$$
$$G^{(n)}(r_1' t_1' \ldots r_n' t_n', r_n' t_n'' \ldots r_1' t_1'') \times \qquad (17.21)$$
$$\prod_{j=1}^{n} \sigma(r_j') S(t_j'' - t_j') dr_j' dt_j' dt_j''.$$

To abbreviate this expression a bit, let us define the function

$$V(x', x'') = \sigma(r') \delta(r' - r'') S(t'' - t'), \qquad (17.22)$$

where x indicates both the position r and the time t. Then the expression for the generating function reduces to

$$Q(\lambda, t) = \sum_{n=1}^{\infty} \frac{(-\lambda)^n}{n!} \int \ldots \int G^{(n)}(x_1' \ldots x_n', x_n'' \ldots x_1'') \times$$
$$\prod_{j=1}^{n} V(x_j' x_j'') d^4 x_j' d^4 x_j''. \qquad (17.23)$$

Since this is a power series expansion about $\lambda = 0$, the factorial moments must be given, according to Eqs. (7.11) and (7.12), by

$$\left\langle \frac{C!}{(C-n)!} \right\rangle = \int \cdots \int G^{(n)}(x_1' \cdots x_n', x_n'' \cdots x_1'') \quad \times$$

$$\prod_{j=1}^{n} V(x_j' x_j'') \ d^4 x_j' d^4 x_j'', \tag{17.24}$$

here the integrations are carried out over the sensitive volume of the counter and the time interval from 0 to t.

As an illustration of the usefulness of these results, let us consider the case of fully coherent field. For such a field we have the factorization

$$G^{(n)}(x_1' \cdots x_n', x_n'' \cdots x_1'') = \prod_{j=1}^{n} G^{(1)}(x_j', x_j''), \tag{17.25}$$

that the series for $Q(\lambda, t)$ may be summed to the form

$$Q(\lambda, t) = e^{-\lambda \int\int G^{(1)}(x',x'') V(x'x'') d^4x' d^4x''}. \tag{17.26}$$

ut from Eq. (17.24) we see that the average number of counts is just

$$\langle C \rangle = \int\int G^{(1)}(x', x'') \ V(x'x'') \ d^4x' \ d^4x''; \tag{17.27}$$

that the generating function may be written as

$$Q(\lambda, t) = e^{-\lambda\langle C\rangle}, \tag{17.28}$$

ow by using Eq. (17.12) we derive the factorial moments

$$\left\langle \frac{C!}{(C-n)!} \right\rangle = \langle C \rangle^n, \tag{17.29}$$

d by using Eq. (17.10) we find that the probability distribution is

$$p(m, t) = \frac{\langle C\rangle^m}{m!} e^{-\langle C\rangle}, \tag{17.30}$$

e., when the field is fully coherent we always have a Poisson distribution for the mber of counts.

When the field does not possess full coherence we can nevertheless use the herent states as a basis for describing it. To illustrate the form the statistical lculations take, we shall use the P-representation for the density operator of e field. The R-representation, which applies more generally, can also be used milarly. In the P-representation $G^{(1)}$ is given by the integral

$$G^{(n)}(x_1 \cdots x_{2n}) = \int P(\{\alpha_k\}) \prod_{j=1}^{n} \mathscr{E}^*(x_j\{\alpha_k\}) \quad \times$$

$$\prod_{j=n+1}^{2n} \mathscr{E}(x_j\{\alpha_k\}) \prod_k d^2\alpha_k. \tag{17.31}$$

en this expression is substituted into the series (17.23) we find that the series y be summed to the closed form

$$Q(\lambda, t) = \int P(\{\alpha_k\}) \ e^{-\lambda\Omega(\{\alpha_k\})} \prod_k d^2\alpha_k, \tag{17.32}$$

ere

$$\Omega(\{\alpha_k\}) = \int \mathscr{E}^*(x'\{\alpha_k\}) \mathscr{E}(x''\{\alpha_k\}) V(x', x'') d^4x' d^4x''. \tag{17.33}$$

rthermore we see from Eq. (17.12) that the factorial moments are

$$\left\langle \frac{C!}{(C-n!)} \right\rangle = \int P(\{\alpha_k\}) \Omega^n(\{\alpha_k\}) \prod_k d^2\alpha_k \tag{17.34}$$

and from Eq. (17.10) that the probability distribution is given by

$$p(m, t) = \int P(\{\alpha_k\}) \frac{\Omega^m(\{\alpha_k\})}{m!} e^{-\Omega(\{\alpha_k\})} \prod_k d^2\alpha_k. \tag{17.35}$$

The probability of counting m photons is evidently a species of average over the corresponding probabilities for an ensemble of Poisson distributions. We hardly need emphasize that the averaging process is not a classical one and that the quasiprobability function P may assume negative values.

As a further illustration of the methods we are discussing let us consider the general case of a chaotically generated field. The density operators of such fields may be represented by means of the Gaussian function

$$P(\{\alpha_k\}) = \prod_k \frac{1}{\pi\langle n_k\rangle} e^{-\frac{|\alpha_k|^2}{\langle n_k\rangle}}. \tag{17.36}$$

Then, since the function Ω is a quadratic form in the variables α_k, it will be possible to evaluate the integral (17.32) for the generating function in full generality.

Before we do this, however, let us introduce some useful notation. We may express the function $\mathscr{E}(x,\{\alpha_k\})$ as a linear form in the variables α_k by using the normal mode expansion

$$\mathscr{E}(x,\{\alpha_k\}) = \sum_k e(x, k)\alpha_k, \tag{17.37}$$

where the functions e are given by Eq. (14.26). If we then define the matrix

$$B_{k'k''} = \int e^*(x'k') V(x'x'') e(x''k'') d^4x' d^4x'', \tag{17.38}$$

we may write the quadratic form Ω as

$$\Omega(\{\alpha_k\}) = \sum_{k'k''} \alpha_{k'}^* B_{k'k''} \alpha_{k''}. \tag{17.39}$$

When this expression for Ω and the Gaussian form for P are substituted in Eq. (17.32) we find that the generating function is given by

$$Q(\lambda, t) = \int \cdots \int \exp\left\{-\sum_k \frac{|\alpha_k|^2}{\langle n_k\rangle} - \lambda \sum_{k'k''} \alpha_{k'}^* B_{k'k''} \alpha_{k''}\right\} \prod_k \frac{d^2\alpha_k}{\pi\langle n_k\rangle}.$$

If we then introduce the variables

$$\beta_k = \alpha_k / \{\langle n_k\rangle\}^{\frac{1}{2}} \tag{17.40}$$

and define the matrix

$$M_{k'k''} = \{\langle n_{k'}\rangle\}^{\frac{1}{2}} B_{k'k''}\{\langle n_{k''}\rangle\}^{\frac{1}{2}}, \tag{17.41}$$

the integral for the generating function may be simplified to the form

$$Q(\lambda, t) = \int \cdots \int \exp\left\{-\sum_k |\beta_k|^2 - \lambda \sum_{k'k''} \beta_{k'}^* M_{k'k''} \beta_{k''}\right\} \prod_k \frac{d^2\beta_k}{\pi} \tag{17.42}$$

Now we can consider the set of numbers β_k as forming the components of a complex vector β. Then if we let M represent the matrix whose components are given by Eq. (17.41), we may write the exponent in the integrand of Eq. (17.42) as the product

$$-\beta^\dagger (1 + \lambda M)\beta.$$

Since the Matrix M is Hermitian it may be diagonalized by carrying out a unitary transformation upon the vector β. Then if we let the eigenvalues of M be $\mathcal{M}_i$, and let the transformed complex coordinates be γ_i, the integral for the generating function reduces to the elementary form

$$Q(\lambda, t) = \int \cdots \int \exp\left\{ - \sum_i (1 + \lambda \mathcal{M}_i) |\gamma_i|^2 \right\} \prod_i \frac{d^2 \gamma_i}{\pi}$$

$$= \frac{1}{\prod_i (1 + \lambda \mathcal{M}_i)} \tag{17.43a}$$

$$= \frac{1}{\det (1 + \lambda M)} \cdot \tag{17.43b}$$

It is worth noting that the matrix M must be positive definite, since the quadratic form Ω defined by Eqs. (17.33) or (17.39) is the average number of photons counted in a particular coherent field. Hence the eigenvalues $\mathcal{M}_i$ are positive, and the singularities of the generating function lie on the negative real axis of the variable λ. Since Q is analytic in the half-plane $\mathrm{Re}\lambda \geq 0$, we see that if we are given Q as a power series expansion about either of the points $\lambda = 0$ or $\lambda = 1$, the series expansion about the other of the points may be evaluated, in principle by analytic continuation. This argument shows that the procedure we have been using, of evaluating the generating function by means of its expansion about $\lambda = 0$, actually leads to a unique answer for the probability distribution.

Since the matrix M is in general of infinite rank, neither of the expressions (17.43) is easy to evaluate directly. Let us note, however, that $\det (1 + \lambda M)$ may be written as

$$\prod_i (1 + \lambda \mathcal{M}_i) = \exp\left\{ \sum_i \log (1 + \lambda \mathcal{M}_i) \right\} .$$

Now for $|\lambda| < (\mathcal{M}_{max})^{-1}$, where $\mathcal{M}_{max}$ is the largest of the eigenvalues $\mathcal{M}_i$, we may expand the logarithm in the exponent in a convergent power series. In this way we see that

$$\det (1 + \lambda M) = \exp\left\{ \sum_i (\lambda \mathcal{M}_i - \frac{1}{2} \lambda^2 \mathcal{M}_i^2 + \cdots) \right\}$$

$$= \exp\left\{ \mathrm{Tr} (\lambda M - \frac{1}{2} \lambda^2 M^2 + \cdots) \right\}$$

$$= \exp\left\{ \mathrm{Tr} \log (1 + \lambda M) \right\} \tag{17.44}$$

where Tr, as always, stands for the trace. By making use of this identity we can express the generating function as

$$Q(\lambda, t) = e^{-\mathrm{Tr} \log(1+\lambda M)} \tag{17.45}$$

If we expand the logarithm in powers of λ, we may write this function in the form

$$Q(\lambda, t) = \exp\left\{ \sum_{r=1}^{\infty} \frac{(-\lambda)^r}{r} I_r \right\} , \tag{17.46}$$

where I_r is defined by

$$I_r = \mathrm{Tr}\{M^r\} . \tag{17.47}$$

If we recall the definition of the matrix M given by Eqs. (17.41) and (17.38), then we see that for $r = 1$ we have

$$I_1 = \iint \sum_k e^*(x' k) \, e(x'' k) \, \langle n_k \rangle \, V(x' x'') \, d^4 x' \, d^4 x'' .$$

The sum over k in the integrand, according to Eq. (14.32), is simply the first order correlation function. The integral thus reduces to

$$I_1 = \iint G^{(1)}(x', x'') \, V(x' x'') \, d^4 x' \, d^4 x'' . \tag{17.48}$$

If we compare Eq. (17.46) with Eqs. (14.27) and (14.28) we see that this $r = 1$ term is of the same form as the exponent of the generating function for the case of a pure coherent field. The lack of coherence for the Gaussian case is reflected by the presence in the exponent of the additional terms with $r \geq 2$. By making further use of the matrix M we can show that the general expression for I_r is the cyclic integral

$$I_r = \int \prod_{j=1}^{r} G^{(1)}(x_j', x_{j+1}'') \, V(x_j' x_j'') \, d^4 x_j' \, d^4 x_j'' , \tag{17.49}$$

in which the coordinate x_{r+1}'' is to be interpreted as x_1''. For the case of broadband detectors the definitions (17.22) and (4.14) allow us to simplify this integral to the form

$$I_r = s^r \int_0^t \cdots \int_0^t \prod_j dt_j' \int \cdots \int \prod_j G^{(1)}(r_j' t_j', r_{j+1}' t_{j+1}) \, \sigma(r_j') \, dr_j' . \tag{17.50}$$

To discuss the evaluation of these integrals let us suppose that our counting experiment has particularly simple geometry. We shall assume that our field consists of plane waves travelling in the positive y-direction, so that the first order correlation function is given by Eq. (15.1). This function naturally depends only on the y-coordinates of its spatial arguments. We next assume that the sensitive region of the counter, i.e., its photocathode, is a very thin layer of atoms lying in a plane perpendicular to the y-axis. The function $\sigma(r)$, in other words, is essentially a delta function of the y-coordinate. With these assumptions, which experiments often approximate quite closely in practice, the spatial integrations in Eq. (17.50) become trivial. The functions $G^{(1)}$ are independent of their position variables for all of the points for which $\sigma(r)$ differs from zero.

The time integrals in Eq. (17.50) are considerably less trivial, but we may discuss the forms they take for short times and for long times. If the time t is much smaller than the inverse frequency bandwidth of the radiation present, the functions $G^{(1)}$ will hardly vary at all in the interval from 0 to t. For such times the integral I_r must simply be proportional to t^r. If we write I_1 as wt, where w is a proportionality constant, then the elementary character of the spatial integrations shows that the general result must be

$$I_r = (wt)^r . \tag{17.51}$$

When this result is substituted in Eq. (17.46), we find that the generating function for small values of t is

$$Q(\lambda, t) = \exp\left\{ -\log (1 + \lambda wt) \right\}$$

$$= \frac{1}{1 + \lambda wt} . \tag{17.52}$$

The probability distribution for the number of counts is then given, according to Eq. (17.10), by

$$p(m, t) = \frac{(wt)^m}{(1 + wt)^{m+1}} . \tag{17.53}$$

The distribution for short times is thus given by a power law not unlike the Planck distribution. The mean number of counts is wt, so that w is simply the average counting rate.

For times t which considerably exceed the inverse bandwith of the radiation field, it is also possible to simplify the integrals I_r. In this case, however, their values depend sensitively on the spectral distribution of the energy present in the field. Let us therefore assume, as an example, that the frequency spectrum has the Lorentz form

$$\langle n_k \rangle \hbar \omega_k = \frac{\text{constant}}{(\omega - \omega_0)^2 + \gamma^2} . \qquad (17.54)$$

The time dependence of the first order correlation function is then given by Eq. 15.8). When this function is substituted into the integral (17.50), we see that, because of the cyclical structure of the integrand, all of the I_r will increase linearly with time for $t \gg \gamma^{-1}$. We may again define the average counting rate, w, by writing the integral I_1 as wt. Then it is not difficult to show that the full set of integrals I_r may be written in the form

$$I_r = \frac{t(2\gamma w)^r}{2(r-1)!} \left(-\frac{1}{2\gamma} \frac{d}{d\gamma} \right)^{r-1} \frac{1}{\gamma} \qquad (17.55)$$

or $t \gg \gamma^{-1}$.

With these values for the I_r it is possible to sum the series in the exponent of Eq. (17.46) in closed form. When this is done we find that the generating function is

$$Q(\lambda, t) = \exp\left\{ -[(\gamma^2 + 2\gamma w\lambda)^{\frac{1}{2}} - \gamma]t \right\} . \qquad (17.56)$$

When the counting rate w is small compared to the frequency bandwidth i.e., $w \ll \gamma$, then the expression in the exponent may be expanded, and we find that in the lowest approximation the generating function reduces to

$$Q(\lambda, t) = e^{-\lambda wt}. \qquad (17.57)$$

This function, as we have seen, leads to a Poisson distribution. It is the distribution we would find if there were no tendency for the photons to arrive in correlated bunches, or for the field amplitude to fluctuate randomly.

To discuss the distribution and moments which follow from the generating function (17.5), it is useful to introduce the set of inverse polynomials

$$s_0(\xi) = s_1(\xi) = 1$$
$$s_2(\xi) = 1 + \frac{1}{\xi}$$
$$s_3(\xi) = 1 + \frac{3}{\xi} + \frac{3}{\xi^2} \qquad (17.58)$$
$$s_4(\xi) = 1 + \frac{6}{\xi} + \frac{5}{\xi^2} + \frac{15}{\xi^3}$$

The further members of the sequence are given by the recursion formula

$$s_{n+1}(\xi) = -s_n'(\xi) + \left(1 + \frac{n}{\xi} \right) s_n(\xi) . \qquad (17.59)$$

These polynomials are quite familiar in the theory of Bessel functions. They may also be calculated from the expression

$$s_n(\xi) = e^\xi \left(\frac{2\xi}{\pi} \right)^{\frac{1}{2}} K_{n-1/2}(\xi) , \qquad (17.60)$$

where $K_{n-1/2}$ is a modified Hankel function of half-integral order.

If we now expand the generating function (17.56) in a power series about $\lambda = 1$ and examine its coefficients we find that the probability of receiving m counts in time t is

$$p(m, t) = \frac{1}{m!} \left(\frac{\gamma wt}{\Gamma} \right)^m s_m(\Gamma t) e^{-(\Gamma - \gamma)t} , \qquad (17.61)$$

where we have written

$$\Gamma = (\gamma^2 + 2w\gamma)^{\frac{1}{2}}. \qquad (17.62)$$

The distribution (17.61) has the same mean value, wt, as the Poisson distribution which follows from the generating function (17.57). Its variance, however, is always larger than that of the Poisson distribution because of the photon clumping effect.

The power series expansion of the generating function (17.56) about $\lambda = 0$ is

$$Q(\lambda t) = \sum_{n=0}^{\infty} \frac{(-\lambda wt)^n}{n!} s_n(\gamma t). \qquad (17.63)$$

We conclude from this expansion that the factorial moments of the distribution (17.61) are given by

$$\left\langle \frac{C!}{(C-n)!} \right\rangle = (wt)^n s_n(\gamma t)$$
$$= \langle C(C-1) \cdots (C-n+1) \rangle. \qquad (17.64)$$

For a Poisson distribution these moments would be simply $(wt)^n$. The first two of the moments (17.64) are

$$\langle C \rangle = wt \qquad (17.65)$$

$$\langle C(C-1) \rangle = (wt)^2 \left(1 + \frac{1}{\gamma t} \right) . \qquad (17.66)$$

The variance of the number of counts is thus

$$\langle C^2 \rangle - \langle C \rangle^2 = \langle C \rangle \left\{ 1 + \frac{\langle C \rangle}{\gamma t} \right\} \qquad (17.67)$$

The term $\langle C \rangle^2 / \gamma t$ is the addition to the variance which is due to the fact that the photon arrival times are not statistically independent of one another.